Proceedings of IAM, V.14, N.1, 2025, pp. 94-104
DOI: 10.30546/2225-0530.14.1.2025.094

MARCINKIEWICZ INTEGRAL AND THEIR COMMUTATORS ON MIXED LEBESGUE SPACES

F.A. Isayev[1], M.N. Omarova[2]

[1]Institute of Mathematics and Mechanics, Ministry of Science and Educations of the Republic of Azerbaijan, Baku, Azerbaijan
[2]Baku State University, Baku, Azerbaijan
e-mail: isayevfatai@yahoo.com, mehriban_omarova@yahoo.com

Abstract. In this paper, we study the boundedness of the Marcinkiewicz operators μ_Ω and their commutators $\mu_{b,\Omega}$ on mixed Lebesgue spaces $L^{\vec{p}}\left(R^n\right)$.

Keywords: Mixed Lebesgue spaces, Marcinkiewicz operator, commutators, BMO.

AMS Subject Classification: 42B20, 42B25, 35J10.

1. Introduction

For $x \in R^n$, and $r > 0$, let $B(x,r)$ be the open ball centered at x with the Radius r, and $B^c(x,r)$ be its complement. The well-known classical Hardy-Littlewood maximal operatör M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $f \in L^1_{loc}\left(R^n\right)$ and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

Let T is a sublinear operator, and satisfies that for any $f \in L^1\left(R^n\right)$ with compact support and $x \notin su\ pp\ f$,

$$|Tf(x)| \le C \int_{R^n} \frac{|f(y)|}{|x-y|^n}\, dy. \tag{1}$$

We point out that the condition (1) was first introduced by Soria and Weiss [20]. The condition (1) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, the Carleson's maximal operators, the Fefferman's singular integrals, the Ricci-Stein's oscillatory singular integrals, the Bochner-Riesz means and so on (see [17, 20] for details).

As is well known, the commutator is also an important operator and it plays a key role in harmonic analysis. Recall that for a locally integrable function b and a integral operatör T, the commutator formed by b and T is defined by $[b,T] = bT - Tb$. The commutators of the fractional maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator have been intensively studied, see for more details. In this paper, the maximal commutator operatör M_b under consideration is of the form

$$M_b f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x) - b(y)| \, |f(y)| \, dy,$$

for $f \in L^1_{loc}(R^n)$.

To study a class of commutators uniformly, one can also introduce some sublinear operators with additional size conditions as before. For a function b, suppose that the commutator operator T_b represents a linear or a sublinear operator, which satisfies that for any $f \in L^1(R^n)$ with compact support and $x \notin supp\, f$,

$$|T_b f(x)| \lesssim \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x-y|^n} |f(y)| dy.$$

The operator T_b has been studied in [12, 17].

Let $S^{n-1} = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n $(n \geq 2)$ equipped with the normalized Lebesgue measure. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on R^n. That is,

$$\Omega(tx) = \Omega(x) \tag{2}$$

for all $t > 0$ and $x \in R^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x') dx' = 0, \tag{3}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. In this paper, we will also consider the commutator $\mu_{\Omega,b}$ which is given by the following expression

$$\mu_{\Omega,b} f(x) = \left(\int_0^\infty \left| F_{\Omega,t}^b(f)(x) \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}^b(f)(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y) dy .$$

On the other hand, the study of Schrödinger operatör $L = -\Delta + V$ recently attracted much attention. In particular, Shen [21] considered L_p estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R_j^L = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \dots, n$. Then, Dziubanński and Zienkiewicz [10] introduced the Hardy type space $H_L^1(R^n)$ associated with the Schrödinger operatör L, which is larger than the classical Hardy space $H^1(R^n)$, see also [1, 2, 3, 13, 14].

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions $\mu_{j,\Omega}$ associated with the Schrödinger operator L by

$$\mu_{j,\Omega}^L f(x) = \left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)| K_j^L(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $K_j^L(x,y) = \tilde{K}_j^L(x,y)|x-y|$ and $\tilde{K}_j^L(x,y)$ is the kernel of $R_j^L = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \dots, n$. In particular, when $V = 0$, $K_j^\Delta(x,y) = \tilde{K}_j^\Delta(x,y)|x-y| = \frac{(x-y)_j / |x-y|}{|x-y|^{n-1}}$ and $\tilde{K}_j^\Delta(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} \Delta^{-\frac{1}{2}}$, $j = 1, \dots, n$. In this paper, we write $K_j(x,y) = K_j^\Delta(x,y)$ and

$$\mu_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)| K_j(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right).$$

Obviously, $\mu_{j,\Omega}$ are classical Marcinkiewicz functions with rough kernel. Therefore, it will be an interesting thing to study the property of $\mu_{j,\Omega}^L$. The main purpose of this paper is to show that Marcinkiewicz operators with rough kernel associated with Schrödinger operators $\mu_{j,\Omega}^{L}$, $j=1,\dots,n$ are bounded on mixed Lebesgue space $L^{\vec{p}}(R^n)$, $1<\vec{p}<\infty$.

The commutator of the classical Marcinkiewicz function with rough kernel is defined by

$$\mu_{j,\Omega,b}\, f(x)=\left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)|\, K_j(x,y)[b(x)-b(y)]f(y)dy \right|^2 \frac{dt}{t^3}\right)^{1/2}$$

The commutator $\mu_{j,\Omega,b}^{L}$ formed by $b\in BMO(R^n)$ and the Marcinkiewicz function with rough kernel $\mu_{j,\Omega}^{L}$ is defined by

$$\mu_{j,\Omega,b}^{L}\, f(x)=\left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)|\, K_j^{L}(x,y)[b(x)-b(y)]f(y)dy \right|^2 \frac{dt}{t^3}\right)^{1/2}$$

Let $f\in L_{loc}^{1}(R^n)$. The maximal operator with rough kernel M_Ω is defined by

$$M_\Omega f(x)=\sup_{t>0}|B(x,t)|^{-1}\int_{B(x,t)}|\Omega(x-y)|\,|f(y)|dy\,.$$

It is obvious that when $\Omega\equiv 1$, M_Ω is the Hardy-Littlewood maximal operator M. For $b\in L_1^{loc}(R^n)$ the commutator of the maximal operator $M_{\Omega,b}$ is defined by

$$M_{\Omega,b} f(x)=\sup_{t>0}|B(x,t)|^{-1}\int_{B(x,t)}|b(x)-b(y)||\Omega(x-y)|\,|f(y)|dy$$

We find the conditions with $b\in BMO(R^n)$ which ensures the boundedness of the operators $\mu_{j,\Omega,b}^{L}$, $j=1,\dots,n$ on mixed Lebesgue space $L^{\vec{p}}(R^n)$, $1<\vec{p}<\infty$.

By A ≲ B, we mean that $A\le CB$ for some constant $C>0$, and $A\approx B$ means that A ≲ B and B ≲ A.

2. Definitions and preliminaries.

Throughout the paper, we use the following notations.

For any $r>0$ and $x\in R^n$, let $B(x,r)=\{y:|y-x|<r\}$ be the ball centered at x with radius r. Let $B=\{B(x,r):x\in R^n, r>0\}$ be the set of all such balls. We also use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E

Definition 2.1. For $1 < p < \infty$, a non-negative function $w \in L_{loc}(R^n)$ is said to be an $A_p(R^n)$ weight if

$$[w]_{A_p} = \sup_{B \in B} \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty .$$

A non-negative local integrable function w is said to be an A_1 weight if

$$\frac{1}{|B|} \int_B w(y) dy \le C w(x) \text{a.e. } x \in B$$

for some constant $C > 0$. The infimum of all such C is denoted by $[w]_{A_1}$. We denote A_∞ by the union of all A_p $(1 \le p < \infty)$ functions.

Theorem 2.1. ([9]) Suppose that Ω be satisfies the conditions (2) and $\Omega \in L_\infty(S^{n-1})$. Then for every $1 < p < \infty$ and $w \in A_p(R^n)$, there is a constant C independent of f such that

$$\| M_\Omega(f) \|_{L^{p,w}} \le C \| f \|_{L^{p,w}} .$$

Theorem 2.2. ([4]) Suppose that Ω be satisfies the conditions (2) and $\Omega \in L_\infty(S^{n-1})$. Let also $b \in BMO(R^n)$. Then for every $1 < p < \infty$ and $w \in A_p(R^n)$, there is a constant C independent of f such that

$$\| M_{\Omega,b}(f) \|_{L^{p,w}} \le C \| f \|_{L^{p,w}} .$$

Theorem 2.3. ([7]) Suppose that Ω be satisfies the conditions (2), (3) and $\Omega \in L_\infty(S^{n-1})$. Then for every $1 < p < \infty$ and $w \in A_p(R^n)$, there is a constant C independent of f such that

$$\| \mu_\Omega(f) \|_{L^{p,w}} \le C \| f \|_{L^{p,w}}$$

Theorem 2.4. ([8]) Suppose that Ω be satisfies the conditions (2), (3) and $\Omega \in L_\infty(S^{n-1})$. Let also $b \in BMO(R^n)$. Then for every $1 < \mathrm{p} < \infty$ and $w \in A_p(R^n)$, there is a constant $C > 0$ independent of f such that

$$\| \mu_{\Omega,b}(f) \|_{L^{p,w}} \le C \| f \|_{L^{p,w}} .$$

Note that a nonnegative locally L_q integrable function $V(x)$ on R^n is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V^q(y) dy \right)^{1/q} \le C \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V(y) dy \right) \tag{4}$$

holds for every ball $x \in R^n$ and $r > 0$, where $B(x,r)$ denotes the open ball centered at x with radius r; see [21]. It is worth pointing out that the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only n and the constant C in (4), such that $V \in B_{q+\varepsilon}$. Throughout this paper, we always assume that $0 \neq V \in B_n$.

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces.

In 2019, Nogayama [19] considered a new Morrey space, with the L^p norm replaced by the mixed Lebesgue norm $L^{\vec{p}}(R^n)$, which is call mixed Morrey spaces.

We first recall the definition of mixed Lebesgue space defined in [5].

Let $\vec{p} = (p_1, \ldots, p_n) \in (0,\infty]^n$. Then the mixed Lebesgue norm $\|\cdot\|_{L\vec{p}}$ or $\|\cdot\|_{L^{(p_1,\ldots,p_n)}}$ is defined by

$$\|f\|_{L^{\vec{p}}} = \|f\|_{L^{(p,\ldots,p_n)}} = \left(\int_R \cdots \left(\int_R \left(\int_R |f(x_1, x_2, \ldots, x_n)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \ldots dx_n \right)^{\frac{1}{p_n}}$$

where $f : R^n \to C$ is a measurable function. If $p_j = \infty$ for some $j = \overline{1,n}$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L^{\vec{p}}(R^n) = L^{(p_1,\ldots p_n)}(R^n)$ to be the set of all locally integrable functions f with $\|f\|_{L^{\vec{p}}} < \infty$.

First, we review the definition of $BMO(R^n)$, the bounded mean oscillation space. A function $f \in L^1_{loc}(R^n)$ belongs to the bounded mean oscillation space $BMO(R^n)$ if

$$\|f\|_{BMO} = \sup_{x \in R^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty .$$

If one regards two functions whose difference is a constant as one, then the space $BMO(R^n)$ is a Banach space with respect to norm $\|\cdot\|_{BMO}$. The John-Nirenberg ineuqalitiy for BMO yields that for any $1 < p < \infty$ and $f \in BMO(R^n)$, the BMO norm of f is equivalent to

$$\|f\|_{BMO^q} = \sup_{x\in R^n, r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}\left|f(y)-f_{B(x,r)}\right|^q dy\right)^{1/q}.$$

Recall that for any $\vec{p}=(p_1,\dots,p_n)\in(1,\infty)^n$, the John-Nirenberg inequality for mixed norm space [16] shows that the BMO norm of all $f\in BMO(R^n)$ is also equivalent to

$$\|f\|_{BMO^{\vec{p}}} = \sup_{x\in R^n, r>0}\frac{\left\|\left(f-f_{B(x,r)}\right)\chi_{B(x,r)}\right\|_{L^{\vec{p}}}}{\left\|\chi_{B(x,r)}\right\|_{L^{\vec{p}}}}.$$

The following property for BMO functions is valid.

Lemma 2.1. Let $f\in BMO(R^n)$. Then for all $0<2r<t$, we have

$$\left|f_{B(x,r)}-f_{B(x,t)}\right|\le C\|f\|_{BMO}\ln\frac{t}{r}.$$

As we know, the Hardy-Littlewood maximal operator M is bounded on $L^{\vec{p}}(R^n)$, $1<\vec{p}<\infty$ (see [19]), but there is no complete boundedness results for some other operators on the mixed Lebesgue spaces. To prove the boundedness of some important operators on the mixed Lebesgue space in a uniform way, we will give the extrapolation theorems on mixed Lebesgue spaces, which have their own interest.

The extrapolaton theory on mixed Lebesgue spaces relies on the classical A_p weight (see [11]).

We also need the boundedness of M on mixed norm space $L^{\vec{p}}(R^n)$[19].

Lemma 2.2. For $1<\vec{p}<\infty$, there holds

$$\|Mf\|_{L^{\vec{p}}(R^n)}\le C\|f\|_{L^{\vec{p}}(R^n)}.$$

By F, we mean a family of pair (f,g) of non-negative measurable functions that are not identical to zero. For such a family $S, p>0$ and a weight $w\in A_q$, the expression

$$\int_{R^n} f(x)^p w(x)dx\le C\int_{R^n} g(x)^p w(x)dx,\ (f,g)\in F$$

means that this inequality holds for all pair $(f,g)\in F$ if the left hand side is finite, and the implicated constant depends only on p and A_q.

Now we give the extrapolation theorems on the mixed Lebesgue spaces. The first one is the diagonal extrapolation theorem.

Theorem 2.5. Let $0<p_0<\infty$ and $\vec{p}=(p_1,\dots,p_n)\in(0,\infty)^n$. Let $f,g\in M(R^n)$. Suppose for every $w\in A_1$, we have

$$\int_{R^n} f(x)^{p_0} w(x)dx \le C \int_{R^n} g(x)^{p_0} w(x)dx \ , \ (f,g)\in F \ . \tag{5}$$

Then if, $\vec{p} > p_0$, we have

$$\|f\|_{L^{\vec{p}}(R^n)} \le C\|g\|_{L^{\vec{p}}(R^n)}, \ (f,g)\in F \ .$$

Proof. Without loss of generality, one may assume f is a non-negative function. We use the Rubio de Francia iteration algorithm presented in .

Let $\bar{\bar{p}} = \vec{p}/p_0$ and $\bar{\bar{p}}' = \vec{p}'/p_0$. By the assumptions and Lemma 2.2, the maximal operator is bounded on $L^{\bar{\bar{p}}'}(R^n)$, so there exists a positive constant B such that

$$\|Mf\|_{L^{\bar{\bar{p}}'}(R^n)} \le B\|f\|_{L^{\bar{\bar{p}}'}(R^n)}.$$

For any non-negative function h, define a new operator Rh by

$$Rh(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k B^k}$$

where for $k \ge 1, M^k$ denotes k iterations of the maximal operator, and M^0 is the identity operator. The operator R satisfies

$$h(x) \le Rh(x), \tag{6}$$

$$\|Rh\|_{L^{\bar{\bar{p}}'}} \le 2\|h\|_{L^{\bar{\bar{p}}'}}, \tag{7}$$

$$\|Rh\|_{A_1} \le 2B.$$

The inequality [16] is straight-forward. Since

$$M(Rh) \le \sum_{k=0}^{\infty} \frac{M^{k+1}h}{2^k B^k} \le 2B\sum_{k=1}^{\infty} \frac{M^k h}{2^k B^k} \le 2BRh,$$

the properties [17] and [18] are consequences of Lemma 2 and the definition of A_1. Since the dual of $L^{\bar{\bar{p}}}(R^n)$ is $L^{\bar{\bar{p}}'}(R^n)$, we get

$$\|f\|_{L^{\vec{p}}}^{p_0} = \left\|f^{\,p_0}\right\|_{L^{p}} \le C\sup\left\{\int_{R^n} |f(x)|^{p_0} h(x)dx : \|h\|_{L^{\bar{\bar{p}}'}} \le 1, h \ge 0\right\}. \tag{8}$$

By Hölder's inequality on the mixed Lebesgue spaces and [16], we have

$$\int_{R^n} f(x)^{p_0} h(x)dx \le C\int_{R^n} f(x)^{p_0} Rh(x)dx \le C\left\|f^{\,p_0}\right\|_{L^{\bar{\bar{p}}}} \|h\|_{L^{\bar{\bar{p}}'}} < \infty . \tag{9}$$

In view of (6) and $Rh \in A_1$, we use (5) with $w = Rh(x)$ to obtain

$$\int_{R^n} f(x)^{p_0} h(x)dx \le C\int_{R^n} f(x)^{p_0} Rh(x)dx \le C\left(\int_{R^n} g(x)^{p_0}[Rh(x)]dx\right)$$

Combining (7) with (9) and using Hölder's inequality on the mixed Lebesgue spaces again, we arrive at

$$\int_{R^n} f(x)^{p_0} h(x)dx \le C\left\| g^{p_0} \right\|_{L^{\bar{\bar{p}}}} \left\|(Rh)\right\|_{L^{\bar{\bar{p}}'}} \approx \|g\|^{p_0}_{L^{\bar{p}}} \left\|(Rh)\right\|_{L^{\bar{\bar{p}}'}} . \quad (10)$$

Therefore

$$\left\|(Rh)\right\|_{L^{\bar{\bar{p}}'}} = \|Rh\|_{L^{\bar{\bar{p}}'}} \le C\|h\|_{L^{\bar{\bar{p}}'}} . \quad (11)$$

By taking supremum over all $h \in L^{\bar{\bar{p}}}(R^n)$ with $\|h\|_{L^{\bar{\bar{p}}}} \le 1$(8), (10) and (11) give us the desired conclusion (2.17).

We point out that when $n = 2$, there are different versions of the diagonal extrapolation theorem [15] and the off-diagonal extrapolation theorem [22] on mixed Lebesgue spaces, which are different form Theorem 2.5.

By the density of smooth functions with compact support $C_c^\infty(R^n)$ in the mixed Lebesgue space $L^{\vec{p}}(R^n)$, $1 < \vec{p} < \infty$ (see [5]), one can apply Theorem 2.5 to the mapping property of some sublinear operators.

Theorem 2.6. Suppose $0 < p_0 < \vec{p} < \infty$ and T is a sublinear operator such that for every $w \in A_1$,

$$\int_{R^n} |Tf(z)|^{p_0} w(z)dz \le C \int_{R^n} |f(z)|^{p_0} w(z)dz \ , \ f \in C_c^\infty(R^n).$$

Then T can be extended to be a bounded operator on $L^{\vec{p}}(R^n)$.

Proof. By Theorem 2.5, for any $f \in C_c^\infty(R^n)$, we have

$$\|Tf\|_{L^{\vec{p}}} \le C\|g\|_{L^{\vec{p}}} .$$

Since T is a sublinear operator, we have $|T(f) - T(g)| \le |T(f - g)|$, and hence, for any $f, g \in C_c^\infty(R^n)$, we have

$$\|Tf - Tg\|_{L^{\vec{p}}} \le \|T(f - g)\|_{L^{\vec{p}}} \le C\|f - g\|_{L^{\vec{p}}} .$$

Since $C_c^\infty(R^n)$ is dense in $L^{\vec{p}}(R^n)$, the above inequalities guarantee that T can be extended to be a bounded operator on $L^{\vec{p}}(R^n)$.

The following corollary is a consequence of Theorem 2.6 and the weighted boundedness of the corresponding operators.

Corollary 2.1. Let $1 < \vec{p} < \infty$, $b \in BMO(R^n)$, then $M, \mu_\Omega, M_b, \mu_{\Omega,b}$ are all bounded on $L^{\vec{p}}(R^n)$.

Proof. It is well known that $M, \mu_\Omega, M_b, \mu_{\Omega,b}$ are all sublinear operators, and bounded on $L^{p_0}_w(\mathbb{R}^n)$ for arbitrary $1 < p_0 < \infty$ and $w \in A_{p_0}$ (see for example).

Since $A_1 \subset A_p$, Theorem 2.6 implies that $M, \mu_\Omega, M_b, \mu_{\Omega,b}$ are all bounded on $L^{\vec{p}}(R^n)$ for all $p_0 < \vec{p} < \infty$. In view of the arbitrariness of $1 < p_0 < \infty$, $M, \mu_\Omega, M_b, \mu_{\Omega,b}$ are also bounded on $L^{\vec{p}}(R^n)$ for all $p_0 < \vec{p} < \infty$.

3. Conclusion. In this paper we study the boundedness of the Marcinkiewicz operator μ_Ω on mixed Lebesgue spaces $L^{\vec{p}}(R^n)$. As an application, we obtain the boundedness of the commutator of the Marcinkiewicz operator $\mu_{b,\Omega}$ on mixed Lebesgue spaces $L^{\vec{p}}(R^n)$.

4. Acknowledgements. The authors would like to express their gratitude to the referees for his very valuable comments and suggestions.

References

1. Akbulut A., Celik S., Omarova M.N. Fractional maximal operator associated with Schrödinger operator and its commutators on vanishing generalized Morrey spaces, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., V.44, N.1, (2024), pp.3-19.
2. Akbulut A., Guliyev R., Ekincioglu I. Calderon-Zygmund operators associated with Schrödinger operator and their commutators on vanishing generalized Morrey spaces, TWMS J. Pure Appl. Math., V.13, N.2, (2022), pp.144-157.
3. Akbulut A., Omarova M.N., Serbetci A. Generalized local mixed Morrey estimates for linear elliptic systems with discontinuous coefficients, Socar Proceedings, N.1, (2025), pp.136-142.
4. Alvarez J., Bagby R.J., Kurtz D.S., Pérez C. Weighted estimates for commutators of linear operators, Studia Math., V.104, (1993), pp.195-209.
5. Benedek A., Panzone R. The spaces L^P with mixed norm, Duke Math. J., V.28, N.3, (1961), pp.301-324.
6. Cruz-Uribe D.V., Martell J.M., Pérez C. Weights, extrapolation and the theory of Rubio de Francia, V.215, Springer Science & Business Media, (2011).
7. Ding Y., Fan D., Pan Y. Weighted boundedness for a class of rough Marcinkiewicz integrals, Indiana Univ. Math. J., V.48, (1999), pp.1037-1055.
8. Ding Y., Lu S., Yabuta K. On commutators of Marcinkiewicz integrals with rough kernel, J. Math. Anal. Appl., V.275, (2002), pp.60-68.
9. Duoandikoetxea J. Weighted norm inequalities for homogeneous singular integrals, Trans. Amer. Math. Soc., V.336, (1993), pp.869-880.
10. Dziubański J., Zienkiewicz J. Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality, Rev. Mat. Iber., V.15, (1999), pp.279-296.

11. Grafakos L. Fourier analysis, V.249, (2008).
12. Guliyev V.S. Generalized weighted Morrey spaces and higher order commutators of sublinear operators, Eurasian Math. J., V.3, N.3, (2012), pp.33-61.
13. Guliyev V.S., Akbulut A., Celik S. Fractional integral related to Schrödinger operator on vanishing generalized mixed Morrey spaces, Bound. Value Probl., V.2024, N.137.
14. Celik S., Guliyev V.S., Akbulut A. Commutator of fractional integral with Lipschitz functions associated with Schrödinger operator on local generalized mixed Morrey spaces, Open Math., V.22, (2024), 20240082.
15. Ho K.P. Strong maximal operator on mixed-norm spaces, Ann. Univ. Ferrara, V.62, N.2, (2016), pp.275-291.
16. Ho K.P. Mixed norm lebesgue spaces with variable exponents and applications, Riv. Mat. Univ. Parma, V.9, N.1, (2018), pp.21-44.
17. Lu G., Lu S., Yang D. Singular integrals and commutators on homogeneous groups, Anal. Math., V.28, N.2, (2002), pp.103-134.
18. Nogayama T. Boundedness of commutators of fractional integral operators on mixed Morrey spaces, Integral Transforms Spec. Funct., V.30, N.10, (2019), pp.790-816.
19. Nogayama T. Mixed Morrey spaces, Positivity, V.23, V.4, (2019), pp.961-1000.
20. Soria F., Weiss G. A remark on singular integrals and power weights, Indiana Univ. Math. J., V.43, (1994), pp.187-204.
21. Shen Z. L^p estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier (Grenoble), V.45, (1995), pp.513-546.
22. Tan J. Off-diagonal extrapolation on mixed variable Lebesgue spaces and its applications to strong fractional maximal operators, Georgian Math. J., V.27, N.4, (2020), pp.637-647.